An identity involving the imaginary error function erfi x

Nicolás Quesada and Aaron Goldberg

The imaginary error function erfi(x) is customarily defined as $\text{erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dt\ e^{t^2}$. It is related to the error function by erfi(x)= erf(ix)/i, and can be also defined as

$$\frac{1}{\pi} \fint_{-\infty}^{\infty} dy\ \frac{e^{ay^2+2by}}{y} = \frac{2}{\pi} \int_0^{\infty} dy\ e^{ay^2} \frac{\sinh(2by)}{y} = \text{erfi}\left(\frac{b}{\sqrt{-a}}\right) \text{ for } a < 0. \tag{1}$$

In this note we investigate a property of a function related to the imaginary error function.

We can define two functions

$$\phi_0(x) \equiv \frac{\exp\left(-x^2\right)}{\sqrt[4]{\pi/2}}, \quad \phi_1(x) \equiv \mathcal{N}(\alpha)\, \phi_0(x)\, \text{erfi}(\alpha x), \quad \mathcal{N}(\alpha) = \sqrt{\frac{\pi}{2\sin^{-1}\left(\frac{\alpha^2}{2-\alpha^2}\right)}}, \tag{2}$$

whose $\mathscr{L}^2$-orthogonality is seen from their respective parities.

We here show that ϕ_1 is, like ϕ_0, $\mathscr{L}^2$-normalized; to this end consider the integral

$$I = \int_{-\infty}^{\infty} dx\ \phi_1^2(x) = \mathcal{N}^2 \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right) \text{erfi}^2(\alpha x). \tag{3}$$

This integral is, to the best of our knowledge, not currently tabulated in standard computer algebra systems (CAS) like Mathematica or Maple or the Wolfram Functions Site.[1] We use Eq. (1) to write

$$I = \frac{\mathcal{N}^2}{\pi^2} \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-2x^2\right) \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\frac{u^2}{\alpha^2} + 2ux\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{v^2}{\alpha^2} + 2vx\right) \tag{4a}$$

$$= \frac{\mathcal{N}^2}{\pi^2} \sqrt{\frac{2}{\pi}} \fint_{-\infty}^{\infty} \frac{du}{u} \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\frac{u^2}{\alpha^2} - \frac{v^2}{\alpha^2}\right) \int_{-\infty}^{\infty} dx \exp\left(-2x^2 + 2ux + 2vx\right) \tag{4b}$$

$$= \frac{\mathcal{N}^2}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \fint_{-\infty}^{\infty} \frac{dv}{v} \exp\left(-\beta v^2 + uv\right), \tag{4c}$$

where we have liberally switched the order of the integrals and written $\beta = 1/\alpha^2 - 1/2$. We now again use Eq. (1) to find

$$I = \frac{\mathcal{N}^2}{\pi^2} \fint_{-\infty}^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \left(\pi\ \text{erfi}\left(\frac{u}{2\sqrt{\beta}}\right)\right) = \frac{2\mathcal{N}^2}{\pi} \underbrace{\int_0^{\infty} \frac{du}{u} \exp\left(-\beta u^2\right) \text{erfi}\left(\frac{u}{2\sqrt{\beta}}\right)}_{\sin^{-1}\frac{1}{2\beta}} = 1. \tag{5}$$

The last integral can be evaluated using any of the aforementioned CAS for $|\beta| \geq 1/2, \text{Re}[\beta] > 0$ (i.e., $|\alpha| \leq 1$). Eq. (5) shows that the set $\{\phi_0, \phi_1\}$ forms an (incomplete) ortho*normal* set in $\mathscr{L}^2$.

[1] except for $\alpha = 1$ when $\phi_1(x)|_{\alpha=1} = \sqrt[4]{\frac{2}{\pi}} F(x)$ is proportional to the Dawson function $F(x)$.