The normal ordered form of $\left(a+a^{\dagger}\right)^{n}$

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To find the normal ordered form of $(a+a^{\dagger})^{n}$ one uses the Baker-Campbell-Hausdorff formula,

$$e^{t(X+Y)} = e^{tX}\ e^{tY}\ e^{-\frac{t^2}{2}[X,Y]}, \tag{1}$$

to find a generating function. Taking $X=a^{\dagger}$ and $Y=a$ and noticing that $[a,a^{\dagger}]=1$ one obtains

$$e^{t(a+a^{\dagger})} = e^{ta^{\dagger}} e^{ta} e^{\frac{t^2}{2}}. \tag{2}$$

Now we can expand each exponential. The left hand side simply reads

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\left(a+a^{\dagger}\right)^{n}, \tag{3}$$

whereas the right side takes the form

$$\sum_{i=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^{2i+m+k}}{2^i i!m!k!}(a^{\dagger})^m (a)^k. \tag{4}$$

To obtain the desired expression take $n=2i+m+k, (i=\frac{n-m-k}{2})$ and rewrite the last expression

$$\sum_{n=0}^{\infty} t^n \sum_{m=0}^{n}\sum_{k=0}^{n-m}{}' \frac{(a^{\dagger})^m (a)^k}{2^{\frac{n-m-k}{2}}\left(\frac{n-m-k}{2}\right)!m!k!}, \tag{5}$$

where $\sum_{k=0}^{n-m}{}'$ means that k takes only the values that make $n-m-k=2i$ even. Equating powers of t^n one arrives to the desired formula

$$\left(a+a^{\dagger}\right)^{n} = n! \sum_{m=0}^{n}\sum_{k=0}^{n-m}{}' \frac{(a^{\dagger})^m (a)^k}{2^{\frac{n-m-k}{2}}\left(\frac{n-m-k}{2}\right)!m!k!}. \tag{6}$$

The first few nontrivial evaluations yield

$$\left(a+a^{\dagger}\right)^{2} = a^2 + 2a^{\dagger}a + \left(a^{\dagger}\right)^2 + 1. \tag{7a}$$

$$\left(a+a^{\dagger}\right)^{3} = a^3 + 3\left(a^{\dagger}\right)^2 a + 3a^{\dagger}a^2 + \left(a^{\dagger}\right)^3 + 3a^{\dagger} + 3a. \tag{7b}$$

$$\left(a+a^{\dagger}\right)^{4} = a^4 + 4a^{\dagger}a^3 + 6\left(a^{\dagger}\right)^2 a^2 + 4\left(a^{\dagger}\right)^3 a + \left(a^{\dagger}\right)^4 + 12a^{\dagger}a + 6\left(a^{\dagger}\right)^2 + 6a^2 + 3. \tag{7c}$$

To obtain the anti-normal ordered form of $\left(a+a^{\dagger}\right)^{n}$ simply take $X=a$ and $Y=a^{\dagger}$.
This function written in *Mathematica* generates the normal ordered form for arbitrary n

```
NormalOrder[n_Integer] :=
(Sum[(n!/(2^((n - m - k)/2)*((n - m - k)/2)!*k!*m!))*SuperDagger[a]^m ** a^k *
If[EvenQ[n - m - k], 1, 0], {m, 0, n}, {k, 0, n - m}] /. {1 ** a_ -> a, a_ ** 1 -> a})
```